Conservation of linear momentum

$$\vec{F} = \frac{D\vec{P}}{Dt}$$

$\vec{F}$ = resultant force

$\vec{P}$ = linear momentum = $\int_{sys} \vec{V}\, dm$

For a differential mass δm:

$$\delta\vec{F} = \frac{D(\vec{V}\,\delta m)}{Dt} = \delta m \frac{D\vec{V}}{Dt} = \delta m\, \vec{a}$$

Forces acting on a differential element:

1) <u>Body forces</u> (gravity, magnetic forces, etc.)

$\hookrightarrow \delta\vec{F}_b = \delta m\, \vec{g}$

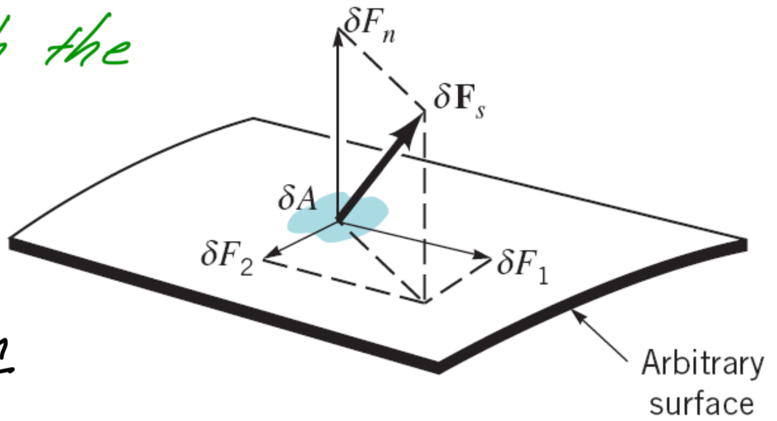
2) <u>Surface forces</u> Interaction with the surroundings

Forces acting on a differential element

1) Body forces (gravity, magnetic forces, etc.)

↳ $\delta \vec{F}_b = \delta m \vec{g}$

2) Surface forces — Interaction with the surroundings

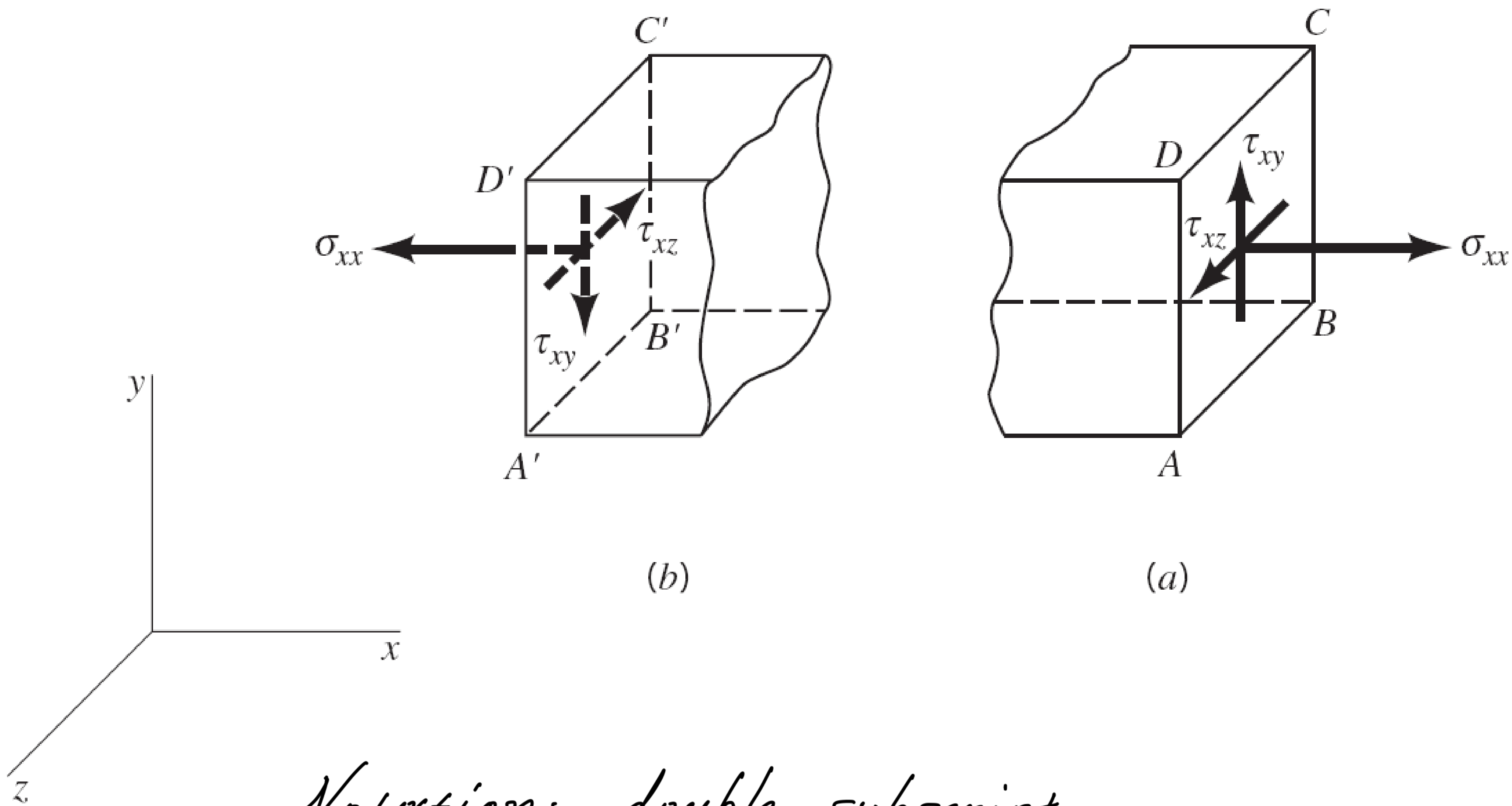


normal stress: $\sigma_n = \lim_{\delta A \to 0} \frac{\delta F_n}{\delta A}$

shear stress: $\tau_1 = \lim_{\delta A \to 0} \frac{\delta F_1}{\delta A}$

$\tau_2 = \lim_{\delta A \to 0} \frac{\delta F_2}{\delta A}$

Double subscript notation for stresses.

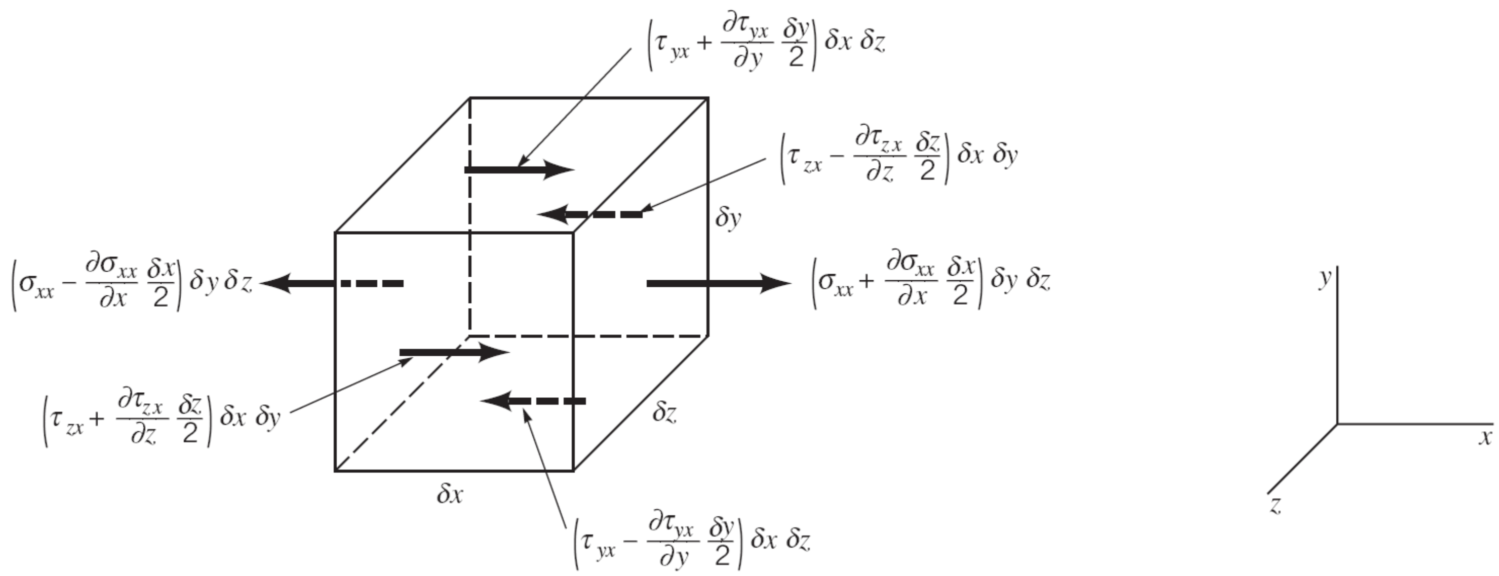


Notation: double subscript

1st subscript: direction of $\perp$ to the surface

2nd subscript: direction of the stress

Surface forces in the x direction acting on a fluid element.



$$\delta F_{S_x} = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)\delta x\,\delta y\,\delta z$$

By analogy,

$$\delta F_{S_y} = \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}\right)\delta x\,\delta y\,\delta z$$

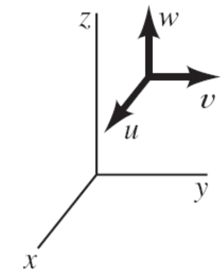
$$\delta F_{S_z} = \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}\right)\delta x\,\delta y\,\delta z$$

Equations of motion

Newton's 2nd law:

$$\delta F_x = \delta m \cdot a_x$$
$$\delta F_y = \delta m \cdot a_y$$
$$\delta F_z = \delta m \cdot a_z$$

where $\delta m = \rho \delta_x \delta_y \delta_z$



x-direction

$$\underbrace{\rho g_x}_{\text{Gravity}} + \underbrace{\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}}_{\text{Surface forces}} = \rho\left(\underbrace{\frac{\partial u}{\partial t}}_{\text{Local Accel.}} + \underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}}_{\text{Convective Acceleration}}\right)$$

y-direction

$$\rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

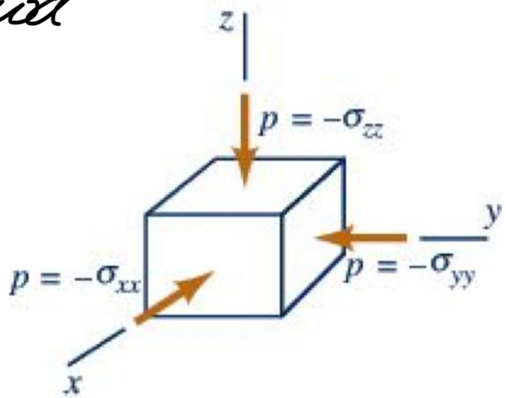
z-direction

$$\rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

Note: these are general equations of motion for <u>solids and fluids</u>

Inviscid flow

- General flow of a fluid is <u>complex</u>
  → need information on shear stress
- In certain cases, viscosity (i.e., air) or viscous effects are small
- If we can <u>neglect</u> viscous forces
  ⇒ <u>inviscid</u> or <u>non-viscous</u> or <u>frictionless</u> flow
- If there are <u>no shearing forces</u>, all normal stresses at a point are equal to $-P$



Euler's equations for inviscid flow

For <u>inviscid flow:</u> $\sigma_{ii} = -p \quad and \quad \tau_{ij} = 0$

Leonhard Euler
(1707-1783)

So, the general equations of motion become:

$$\rho g_x - \frac{\partial p}{\partial x} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y - \frac{\partial p}{\partial y} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z - \frac{\partial p}{\partial z} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

Or, in vector notation:

$$\rho\vec{g} - \vec{\nabla}p = \rho\left[\frac{\partial\vec{V}}{\partial t} + \left(\vec{V}\bullet\vec{\nabla}\right)\vec{V}\right]$$

Irrotational flow

Rotation of a fluid element: $\vec{\omega} = \frac{1}{2} \vec{\nabla} \times \vec{V}$

Vorticity: $\vec{\zeta} = \vec{\nabla} \times \vec{V}$

If flow is irrotational $\rightarrow$ $\vec{\omega} = \vec{\zeta} = 0$

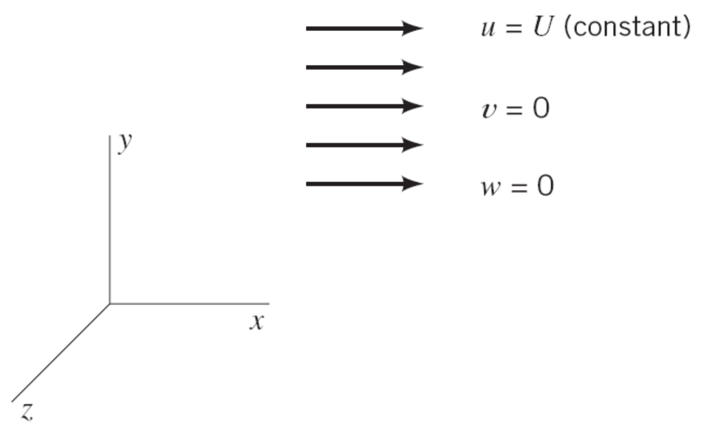
$\rightarrow \vec{\nabla} \times \vec{V} = 0 \rightarrow$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

$$\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0$$

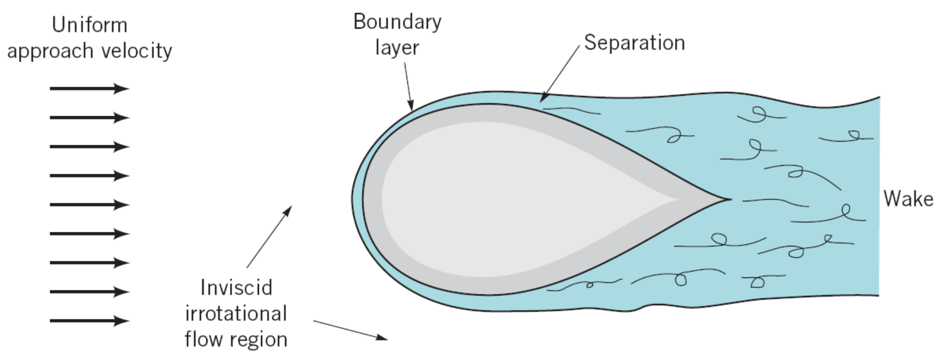
$$\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0$$

Example of irrotational flow:
<u>Uniform flow</u>



Examples of rotational and irrotational flow

Flow around bodies

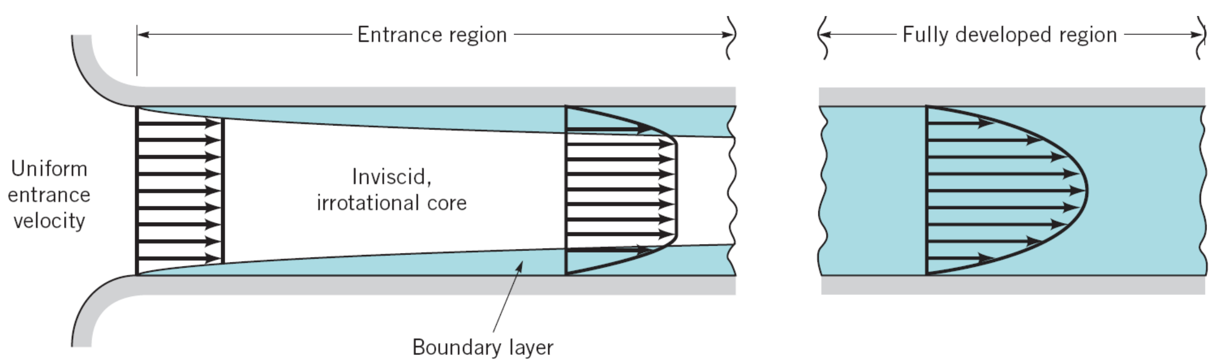


Inviscid region:
→ Only body forces
and pressure

⇓

No rotation
of fluid elements

Flow through channels



In boundary layer:
→ shear forces
→ rotation of fluid
elements

The Velocity Potential

For irrotational flow:

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \quad (1)$$

$$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z} \quad (2)$$

$$\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x} \quad (3)$$

Eqns (1), (2) & (3) are automatically satisfied if we define a scalar function $\phi(x, y, z, t)$ such that:

$$u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y} \qquad w = \frac{\partial \phi}{\partial z}$$

ϕ is called the velocity potential

In vector form: $\vec{\nabla}\phi = \vec{V}$

$$\vec{\nabla}\phi = \frac{\partial \phi}{\partial x}\hat{\imath} + \frac{\partial \phi}{\partial y}\hat{\jmath} + \frac{\partial \phi}{\partial z}\hat{k} = u\hat{\imath} + v\hat{\jmath} + w\hat{k} = \vec{V}$$

Stream Function and Velocity Potential

	Relation to V:	Derives from:	Applies to:
Stream function, ψ	$u = \frac{\partial \psi}{\partial y}$ $v = -\frac{\partial \psi}{\partial x}$	Continuity	2-D flows
Velocity potential, ϕ	$u = \frac{\partial \phi}{\partial x}$ $v = \frac{\partial \phi}{\partial y}$ $w = \frac{\partial \phi}{\partial z}$	Irrotationality	3-D flows

Potential flow

For an incompressible fluid, the conservation of mass is expressed as:

$$\vec{\nabla}\cdot\vec{V} = 0 \quad \text{or} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

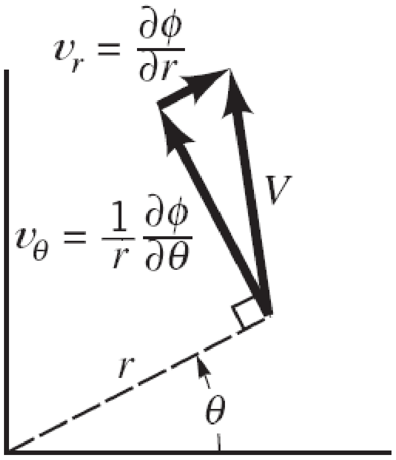
So, for irrotational flow of an incompressible fluid:

$$\left.\begin{array}{l} \vec{\nabla}\cdot(\vec{\nabla}\phi) = 0 \Rightarrow \underline{\underline{\nabla^2\phi = 0}} \\ \text{or } \underline{\underline{\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial^2\phi}{\partial z^2} = 0}} \end{array}\right\} \text{Laplace's equation}$$

<u>Method</u>: Apply Laplace's equation, using boundary conditions, to obtain $\phi(x,y,z,t)$ → derive the velocity field $\vec{V} = \vec{\nabla}\phi$

<u>Important</u>: For irrotational flow, Bernoulli's equation applies to the <u>entire</u> flow field, not only along a streamline.

Velocity potential in cylindrical coordinates



$$\vec{V} = \vec{\nabla}\phi = \frac{\partial \phi}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\hat{e}_\theta + \frac{\partial \phi}{\partial z}\hat{e}_z$$

where

$$v_r = \frac{\partial \phi}{\partial r}$$

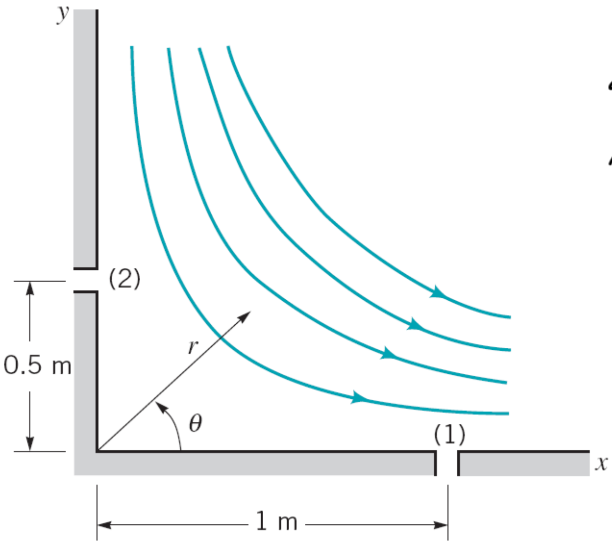
$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

$$v_z = \frac{\partial \phi}{\partial z}$$

Laplace's equation in cylindrical coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

Example on potential flow



2-D flow of a non-viscous incompressible fluid in a 90° corner.

Given:

$\psi = 2r^2 \sin 2\theta$

$P_1 = 30$ kPa

Questions: a) What is the velocity potential?

b) What is pressure at point (2)?

a) Using polar coordinates: $v_r = \dfrac{1}{r}\dfrac{\partial \psi}{\partial \theta} = 4r\cos 2\theta$

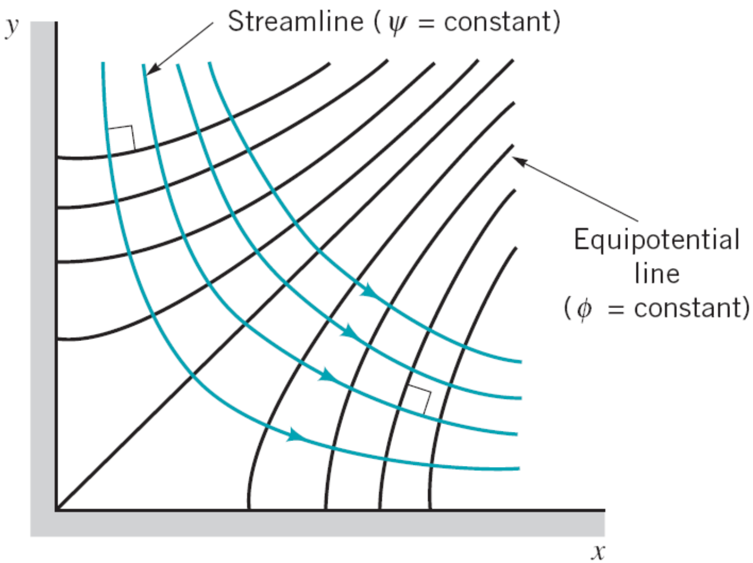
and $v_\theta = -\dfrac{\partial \psi}{\partial r} = -4r\sin 2\theta$

Since $v_r = \dfrac{\partial \phi}{\partial r} = 4r\cos 2\theta \Rightarrow \phi = 2r^2\cos 2\theta + f_1(\theta) \quad (1)$

Also, $v_\theta = \dfrac{1}{r}\dfrac{\partial \phi}{\partial \theta} = -4r\sin 2\theta \Rightarrow \phi = 2r^2\cos 2\theta + f_2(r) \quad (2)$

To satisfy both (1) & (2): $f_1(\theta) = f_2(r) = C \Rightarrow \phi = 2r^2\cos 2\theta + C$

C is arbitrary; chose $C = 0$, so that $\underline{\underline{\Phi = 2r^2\cos 2\theta}}$

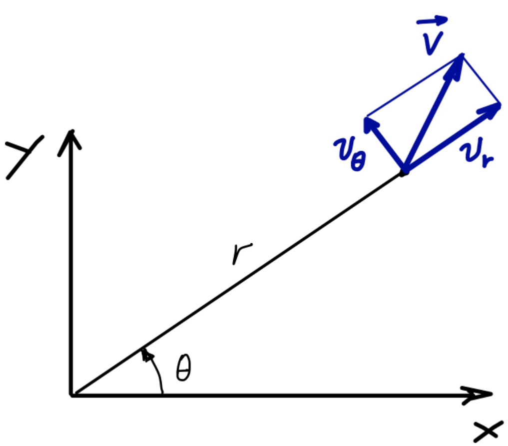


Remark: Streamlines ($\Psi = ct$) and lines of $\Phi = ct$ are always orthogonal to each other

b) Since flow is irrotational, Bernoulli can be applied between any two points in the flow field

$$p_1 + \frac{1}{2}\rho V_1^2 + \cancel{\gamma z_1} = p_2 + \frac{1}{2}\rho V_2^2 + \cancel{\gamma z_2}$$

$$\Rightarrow p_2 = p_1 + \frac{\rho}{2}\left(V_1^2 - V_2^2\right) \quad (3)$$



$$V^2 = v_r^2 + v_\theta^2 = (4r\cos 2\theta)^2 + (-4r\sin 2\theta)^2$$

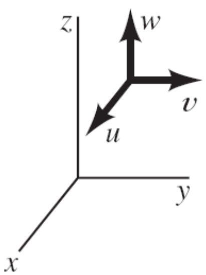
$$= 16r^2(\cos^2 2\theta + \sin^2 2\theta) = 16r^2$$

Hence, $V_1^2 = 16 \times 1^2 \, m^2/s^2 = \underline{16 \, m^2/s^2}$

and $V_2^2 = 16 \times 0.5^2 \, m^2/s^2 = \underline{4 \, m^2/s^2}$

$$(3) \Rightarrow p_2 = 30'000 \, Pa + \frac{1000 \, kg/m^3}{2}\left(16 \, m^2/s^2 - 4 \, m^2/s^2\right) \Rightarrow p_2 = \underline{\underline{36'000 \, Pa}}$$

Viscous flow



Equations of motion in cartesian coordinates:

x-direction
$$\rho(\underbrace{\frac{\partial u}{\partial t}}_{\text{Local Accel.}} + \underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}}_{\text{Convective Acceleration}}) = \underbrace{\rho g_x}_{\text{Gravity}} + \underbrace{(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z})}_{\text{Surface forces}}$$

y-direction
$$\rho(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}) = \rho g_y + (\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z})$$

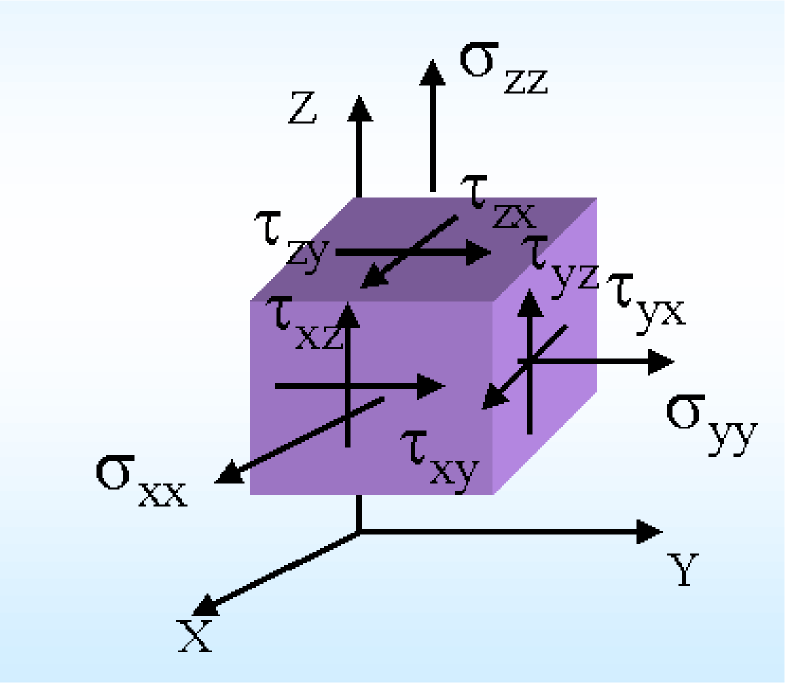
z-direction
$$\rho(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}) = \rho g_z + (\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z})$$

The equations of motion include stresses (σ_{ij}) and velocities (*u*, *v* and *w*)

We need a relationship between stresses (σ_{ij}) and velocities (u, v and w)

Stress-deformation relationships

For Newtonian, incompressible fluids, stresses are linearly related to deformations



Normal stresses:

$$\sigma_{xx} = -p + 2\mu \frac{\partial u}{\partial x}$$

$$\sigma_{yy} = -p + 2\mu \frac{\partial v}{\partial y}$$

$$\sigma_{zz} = -p + 2\mu \frac{\partial w}{\partial z}$$

Shearing stresses:

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

Eliminate stresses

In x-direction: $\rho(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}) = \rho g_x + (\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z})$

$$\sigma_{xx} = -p + 2\mu\frac{\partial u}{\partial x} \longrightarrow \frac{\partial \sigma_{xx}}{\partial x} = -\frac{\partial p}{\partial x} + 2\mu\frac{\partial^2 u}{\partial x^2}$$

$$\tau_{xy} = \tau_{yx} = \mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \longrightarrow \frac{\partial \tau_{yx}}{\partial y} = \mu\frac{\partial^2 u}{\partial y^2} + \mu\frac{\partial^2 v}{\partial x \partial y}$$

$$\tau_{zx} = \tau_{xz} = \mu(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}) \longrightarrow \frac{\partial \tau_{zx}}{\partial z} = \mu\frac{\partial^2 w}{\partial x \partial z} + \mu\frac{\partial^2 u}{\partial z^2}$$

The right hand term of the equation becomes:

$$\rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho g_x - \frac{\partial p}{\partial x} + 2\mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial^2 u}{\partial x^2} + \mu\frac{\partial v}{\partial x \partial y} + \mu\frac{\partial w}{\partial x \partial z} + \mu\frac{\partial^2 u}{\partial z^2}$$

$$= \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \mu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial x \partial y} + \frac{\partial w}{\partial x \partial z})$$

$$= \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \mu\frac{\partial}{\partial x}\underbrace{(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z})}_{=0 \text{ : continuity equation}}$$

$$= \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

Navier-Stokes equations (cartesian)

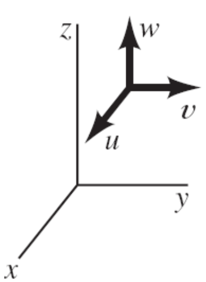
x-direction: $$\underbrace{\rho\left(\frac{\partial u}{\partial t}\right.}_{\text{Local Accel.}} + \underbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\left.\right)}_{\text{Convective Acceleration}} = \underbrace{-\frac{\partial p}{\partial x}}_{\text{Pressure}} + \underbrace{\rho g_x}_{\text{Gravity}} + \underbrace{\mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)}_{\text{Viscous forces}}$$

y-direction: $$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \rho g_y + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

z-direction: $$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \rho g_z + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

Momentum equations

+



$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Continuity equation